Dynamical Models of Cyber-physical Systems (CPSs)

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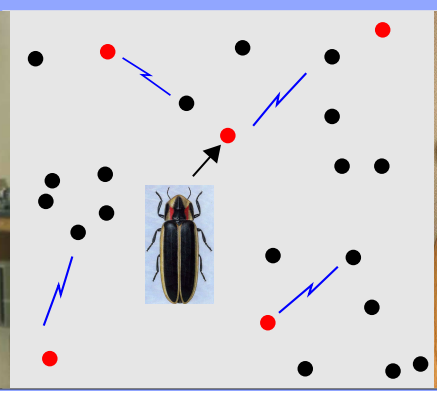

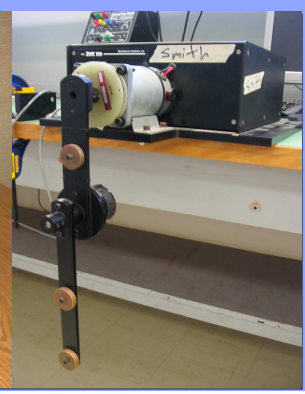

Plan of Work

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- Continuous-time systems

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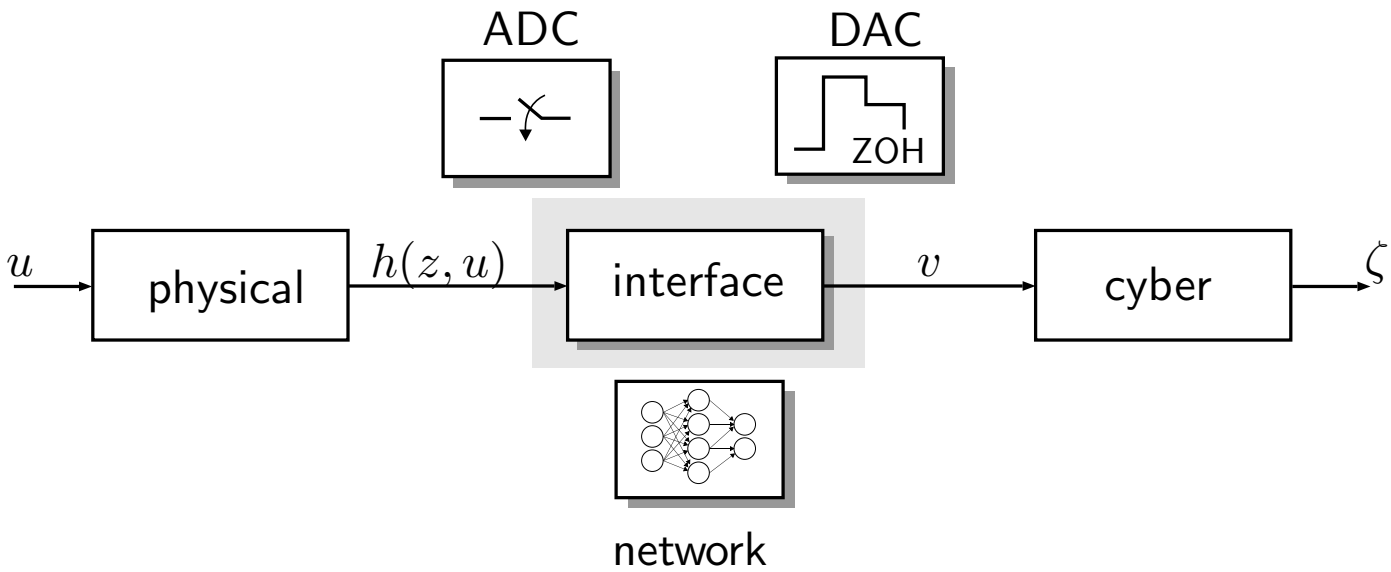
Interconnection Between Physical and Cyber Components



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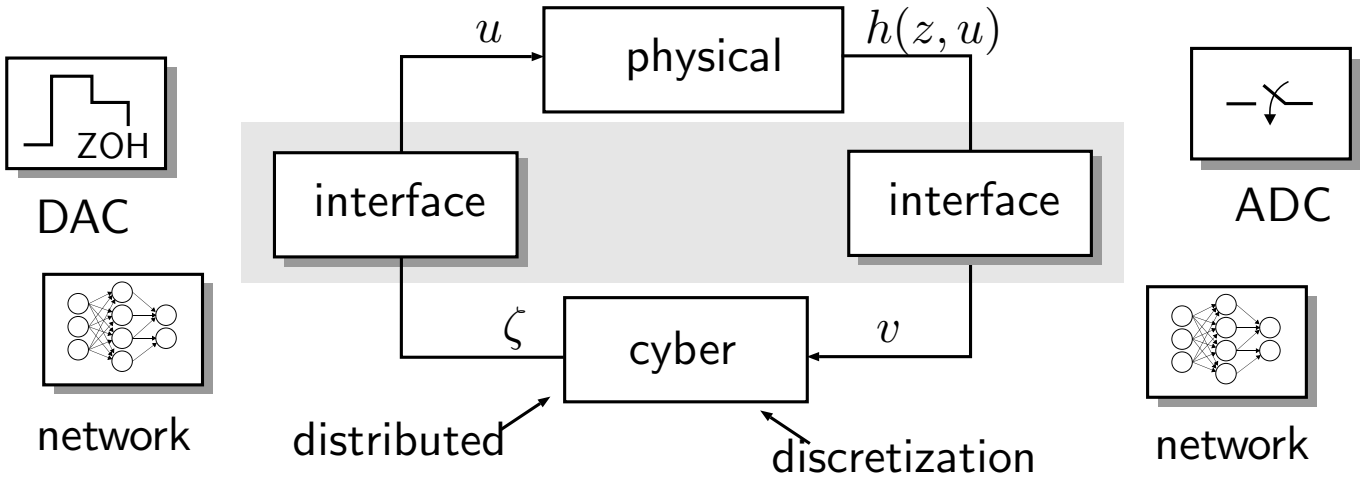

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Outline

1. Models of physical components
2. Models of cyber components
   - Pure finite state machines
   - Finite state machines with conditional structures as guards
   - Models of computer computations and discrete-time algorithms
3. Models of the interface between physical and cyber components
   - Analog-to-Digital converters
   - Digital-to-Analog converters
   - Digital networks
4. Combining models of physical and cyber components
5. Simulating CPSs

Models of Physical Components

The physical components of a cyber-physical system include the analog elements, physical systems, and the environment. Among the many possible models available, we will capture the dynamics of the physical components by differential equations or inclusions.

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The general mathematical description of the physical component is

$$\dot{z} \in F_P(z, u), \quad y = h(z, u)$$

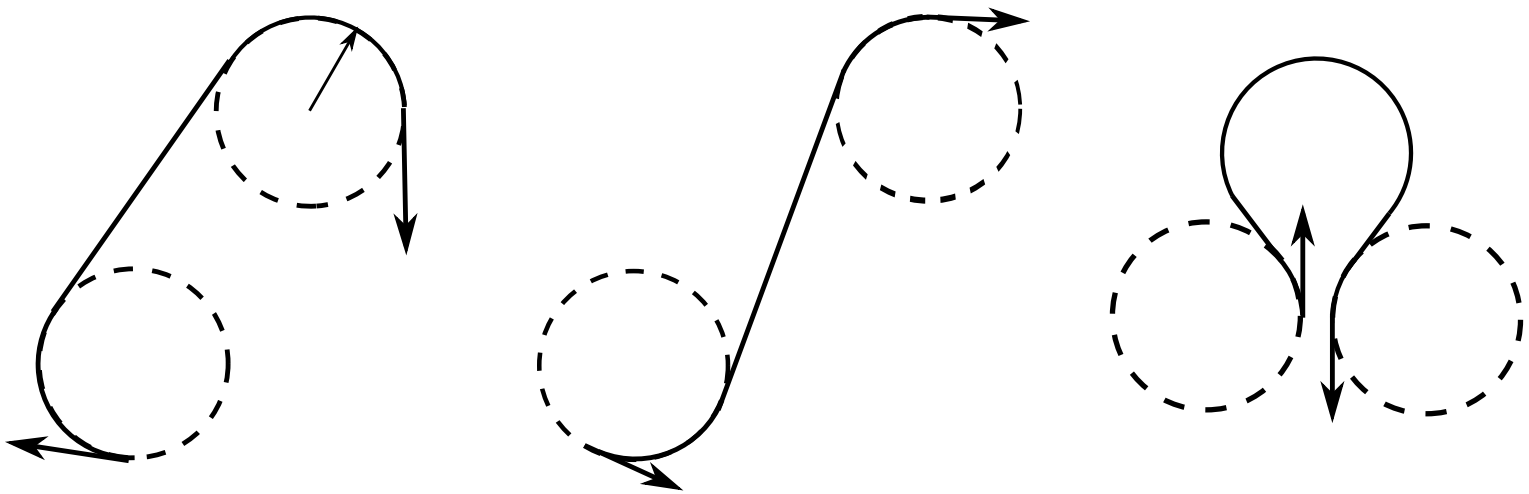
subject to the constraint

$$(z, u) \in C_P \subset \mathbb{R}^{n_P} \times \mathbb{R}^{m_P}$$

An Example of Physical Component

Dubins vehicle

Assumed to be a particle on the plane that describes smooth paths and capable of making turns satisfying a minimum turning radius constraint (similar to a car).



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Denoting its position by $(z_1, z_2) \in \mathbb{R}^2$ and its orientation by $z_3 \in \mathbb{R}$ (with respect to the vertical axis), it can be captured as in

$$F_P(z,u) = \begin{bmatrix} \nu \sin z_3 \\ \nu \cos z_3 \\ u \end{bmatrix}, \qquad C_P = \left\{(z,u) \in \mathbb{R}^3 \times \mathbb{R} \,:\, u \in [-\overline{u}, \overline{u}]\right\}$$

where ν is the velocity of the vehicle, $u \in [-\overline{u}, \overline{u}]$ is the angular velocity input, $\overline{u} = \frac{\nu}{\rho}$, and ρ is the minimum turning radius.

Other Continuous-time Models

Models of Cyber Components

The cyber components of a CPS include those in charge of performing computations, implementing algorithms, and transmitting digital data over networks.

- variables change at discrete events, not necessarily periodically

We capture it by difference equations

The general mathematical description of the cyber component is

$$\eta^+ \in G_C(\eta, v), \quad \zeta = \kappa(\eta, v)$$

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Models of Cyber Components

Widely used models of cyber components

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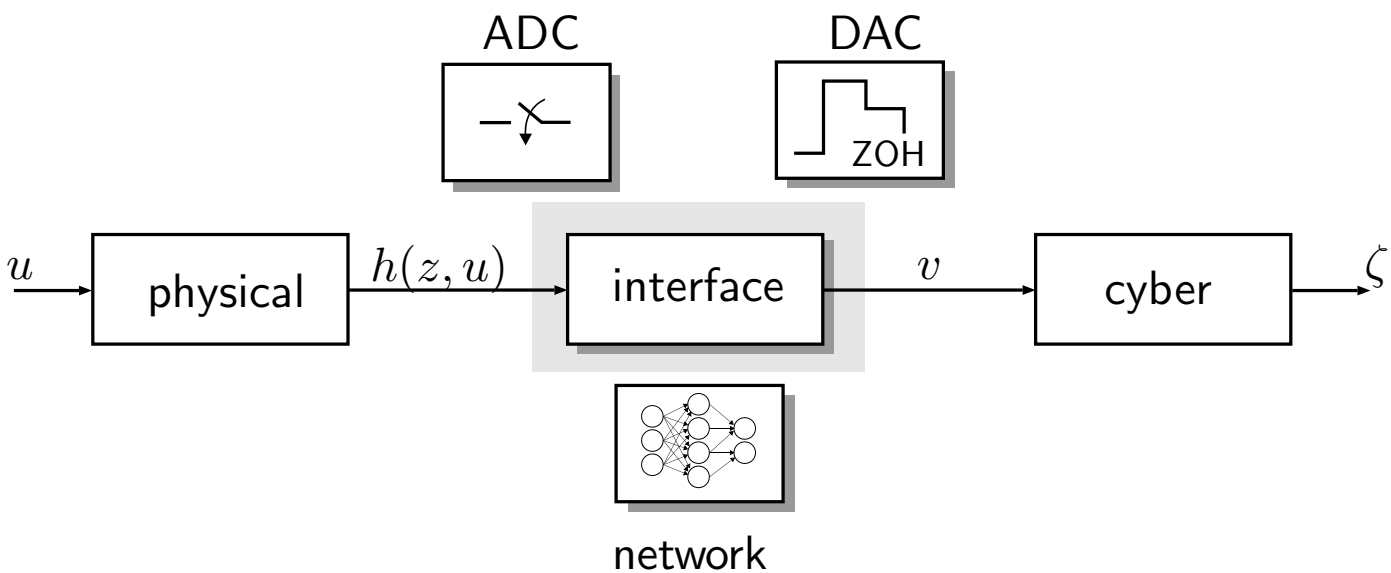

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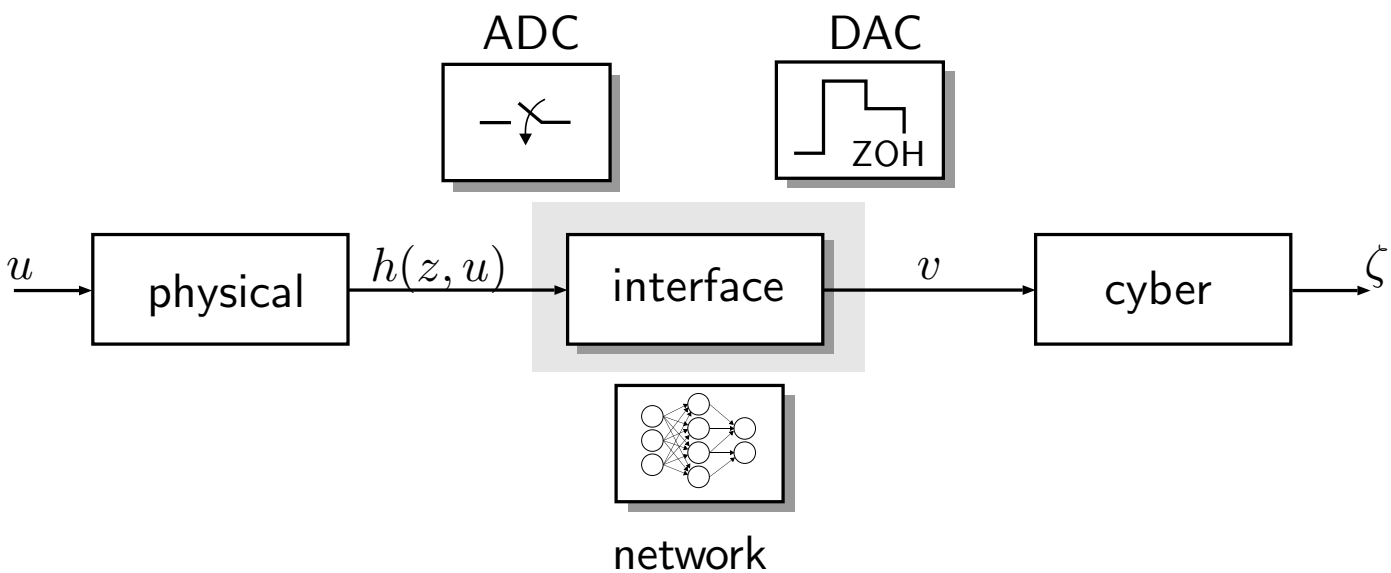

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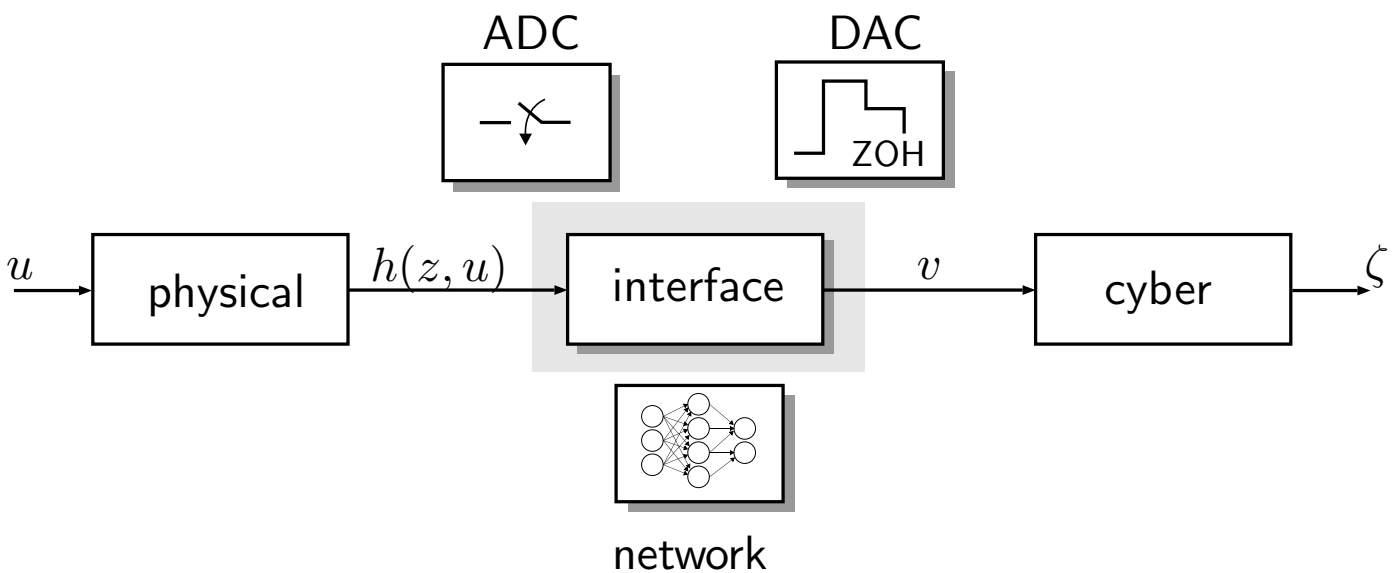
- Discrete-time models
- Pure finite state machines
- Conditional/Triggered finite state machines



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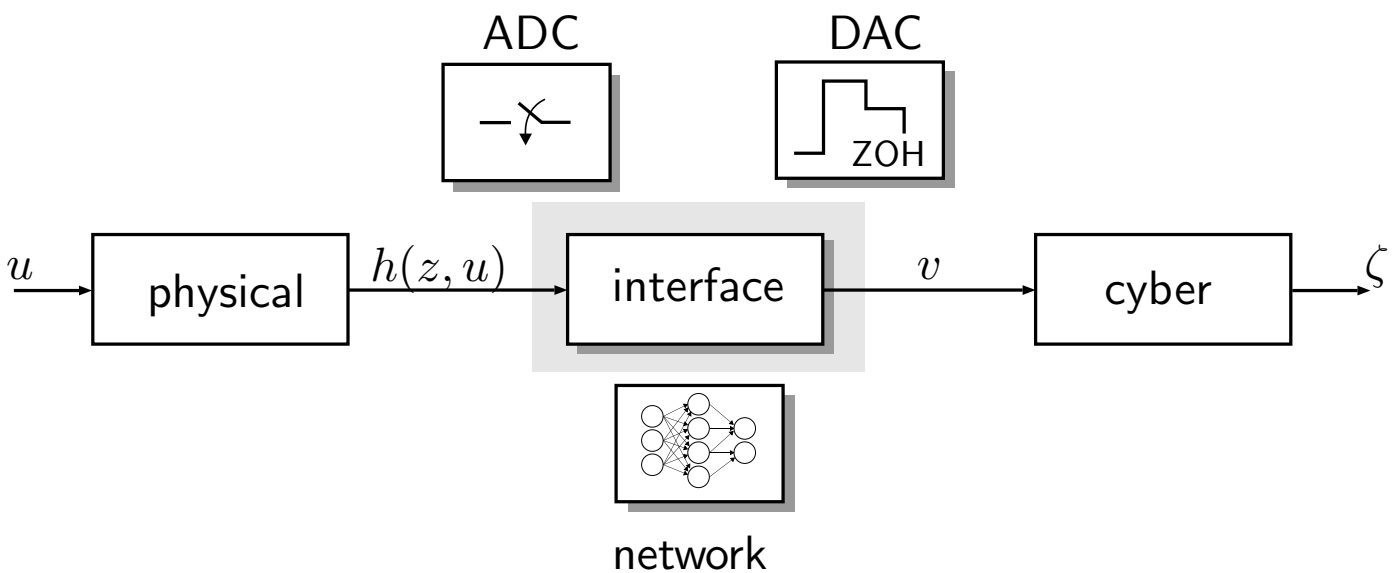

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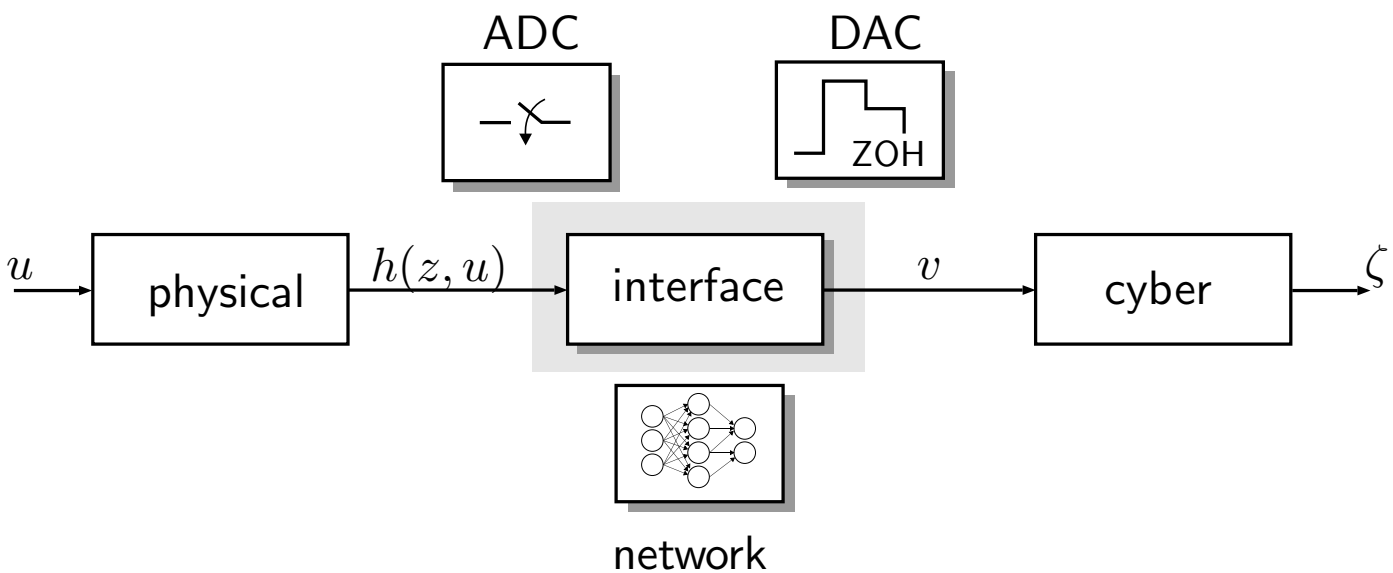

Pure Finite State Machines

A finite state machine (FSM) or deterministic finite automaton (DFA) is a system with inputs, states, and outputs taking values from discrete sets that are updated at discrete transitions (or jumps) triggered by its inputs. Let v denote the inputs, q denote the states (or mode), and r denote the outputs of the FSM.

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The initial state of the FSM is denoted q_0 while, at times, a set of final states is imposed and denoted as $Q_\infty \subset Q$.

$$q^+ = \delta(q, v) \qquad \zeta = \kappa(q) \qquad (q, v) \in Q \times \Sigma$$

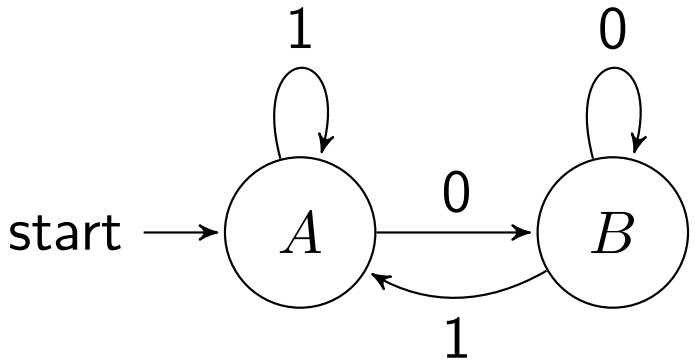
An Example of a FSM

$$Q = \{A, B\}$$

$$\Sigma = \{0, 1\}$$

$$\delta(q, v) = \begin{cases} A & \text{if } v = 1 \\ B & \text{if } v = 0 \end{cases}$$

$$\kappa(q) = q$$



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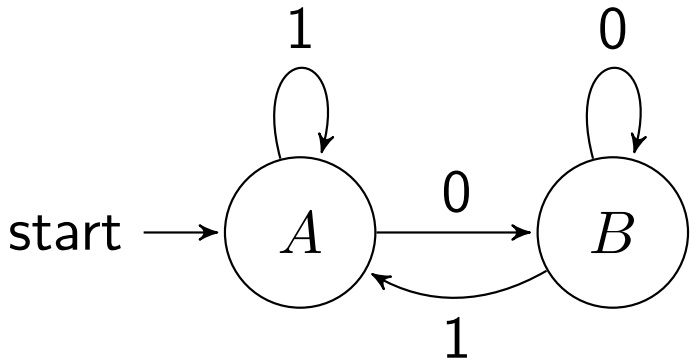

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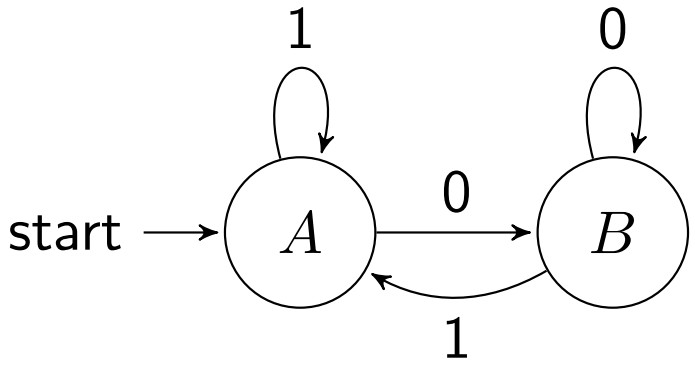

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The nondeterministic case of a FSM can be treated similarly by using a set-valued transition function $\delta : Q \times \Sigma \rightrightarrows Q \times \Delta$.

$$\delta(q, ab) = \bigcup_{q' \in \delta(q,a)} \delta(q', b)$$

for each $a, b \in \Sigma$.

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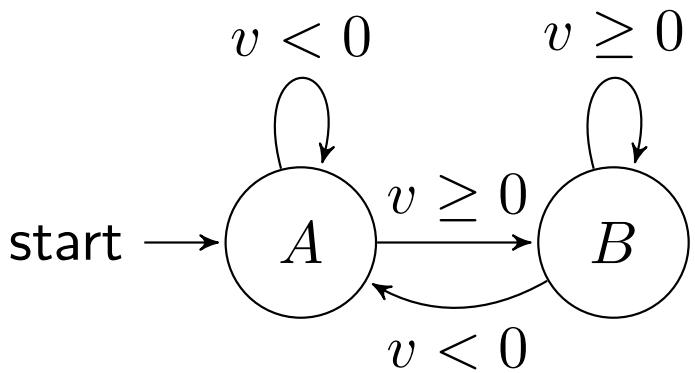
$$q^+ = \delta(q, v) \quad \zeta = \kappa(q) \quad \ell(q, v, \zeta) \leq 0, \quad (q, v) \in Q \times \Sigma$$

An Example of Triggered FSM

Suppose we want to transition when $v < 0$. Since $v \in \Sigma$ which is a finite set (finite alphabet), we need to check $v < 0$ over an infinite alphabet. The technical solution is to use a function Dataflow (DC) to implement:

```
function out = checkCond (v)
if v < 0 then
    out = 1;
else
    out = 0;
end if
return out
```



The output, which is 0 or 1, is feed to the pure FSM.

Models of Computer Computations

Computations and discrete-time algorithms

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- Discrete-time algorithms are naturally modeled using difference equations, which can be captured by

$$\eta^+ = G_C(\eta, v) \qquad \zeta = \kappa(\eta)$$

where G_C is obtained from discretizing the control algorithm.

Iterative Computations

Iterative implementations of computations require a number of steps to reach an answer and, at times, additional variables.

- denoting the additional variables as $m \in \mathbb{R}^{nC-1}$ and the counter as $k \in \{0, 1, 2, \ldots, k^*\}$, $k^* \in \{0, 1, 2, \ldots\} =: \mathbb{N}$,

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then, a model is given by

$$\eta^+ = \begin{bmatrix} \widetilde{\kappa}(m, k, v) \\ k + 1 \end{bmatrix} \qquad \zeta = m,$$

$$k \in \{0, 1, 2, \dots, k^* - 1\}, m \in \mathbb{R}^{n_C - 1}, v \in \mathcal{V}$$

An Example of Iterative Computations

The computation of the factorial of an integer $a > 0$ is given by the evaluation at $v = a$ of the function[1]

$$\widetilde{\kappa}(v) := v(v-1)(v-2)\dots(v-(v-2))1$$

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An Example of Iterative Computations

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The one-step computation of the factorial of the integer a can be performed iteratively by initializing m to a and k to one, setting $k^* = a$, and defining

$$\widetilde{\kappa}(m, k, v) = m(m-k)$$

Note that for $k < k^*$, ζ is equal to the intermediate result $a! - (a-k)!$

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Interfaces need to convert conditions and signals appropriately. Some widely used interfaces include

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- **Digital-to-Analog Converters**: the digital signals in the cyber components need to be converted to analog signals for their use in the physical world
- **Digital Networks**: the information transfer between the physical and cyber components, or between subsystems within the cyber components, might occur over a digital communication network

Analog-to-Digital Converters

A basic model for a sampling device consists of a timer state and a sample state.

When the timer reaches the value of the sampling time T_s^*, the timer is reset to zero and the sample state is updated with the inputs to the sampling device.

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$$\begin{bmatrix} \dot{\tau}_s \\ \dot{m}_s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (\tau_s, m_s, v_s) \in C_s$$

$$\begin{bmatrix} \tau_s^+ \\ m_s^+ \end{bmatrix} = \begin{bmatrix} 0 \\ v_s \end{bmatrix} \qquad (\tau_s, m_s, v_s) \in D_s$$

$$C_s = \{(\tau_s, m_s, v_s) : \tau_s \in [0, T_s^*]\}$$

$$D_s = \{(\tau_s, m_s, v_s) : \tau_s \in [T_s^*, \infty)\}$$

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Omitted effects (can be incorporated if needed):

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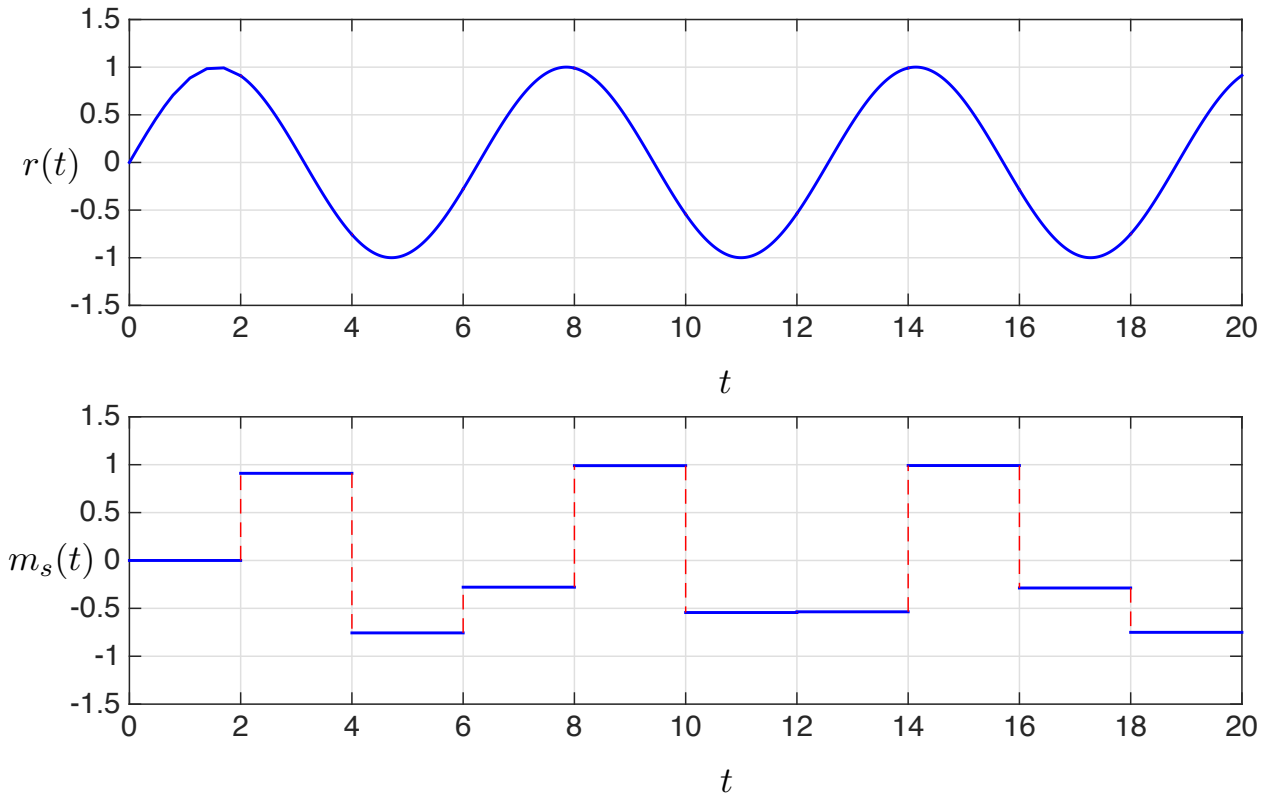
Omitted effects (can be incorporated if needed):

- ADC acquisition time, limits the maximum sampling frequency
- Finite-length store space, which causes quantization

A Simulation of ADC

Parameters: $T^* = 2$, $\tau_s(0,0) = 0$, $m_s(0,0) = 0$.



Writing the ADC Model as a Hybrid Inclusion

Digital-to-Analog Converters

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Let $\tau_h \in \mathbb{R}_{\geq 0}$ be the timer state, $m_h \in \mathbb{R}^{r_C}$ be the sample state, and $v_h \in \mathbb{R}^{r_C}$ be the inputs of the DAC. When $\tau_h \geq T_h^*$, the timer state is reset to zero and the sample state is updated with v_h.

A model is given by

$$\begin{aligned} \dot{\tau}_h = 1, \quad \dot{m}_h = 0 \qquad & \text{when} \quad \tau_h \in [0, T_h^*] \\ \tau_h^+ = 0, \quad m_h^+ = v_h \qquad & \text{when} \quad \tau_h \geq T_h^* \end{aligned}$$

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$$T_N^{*\min} \leq t_{i+1} - t_i \leq T_N^{*\max} \qquad \forall i \in \{1, 2, \ldots, i^* - 1\}$$

where $T_N^{*\min}$ and $T_N^{*\max}$ are constants satisfying

$$T_N^{*\min}, T_N^{*\max} \in (0, \infty]$$

and

$$T_N^{*\min} \leq T_N^{*\max}$$

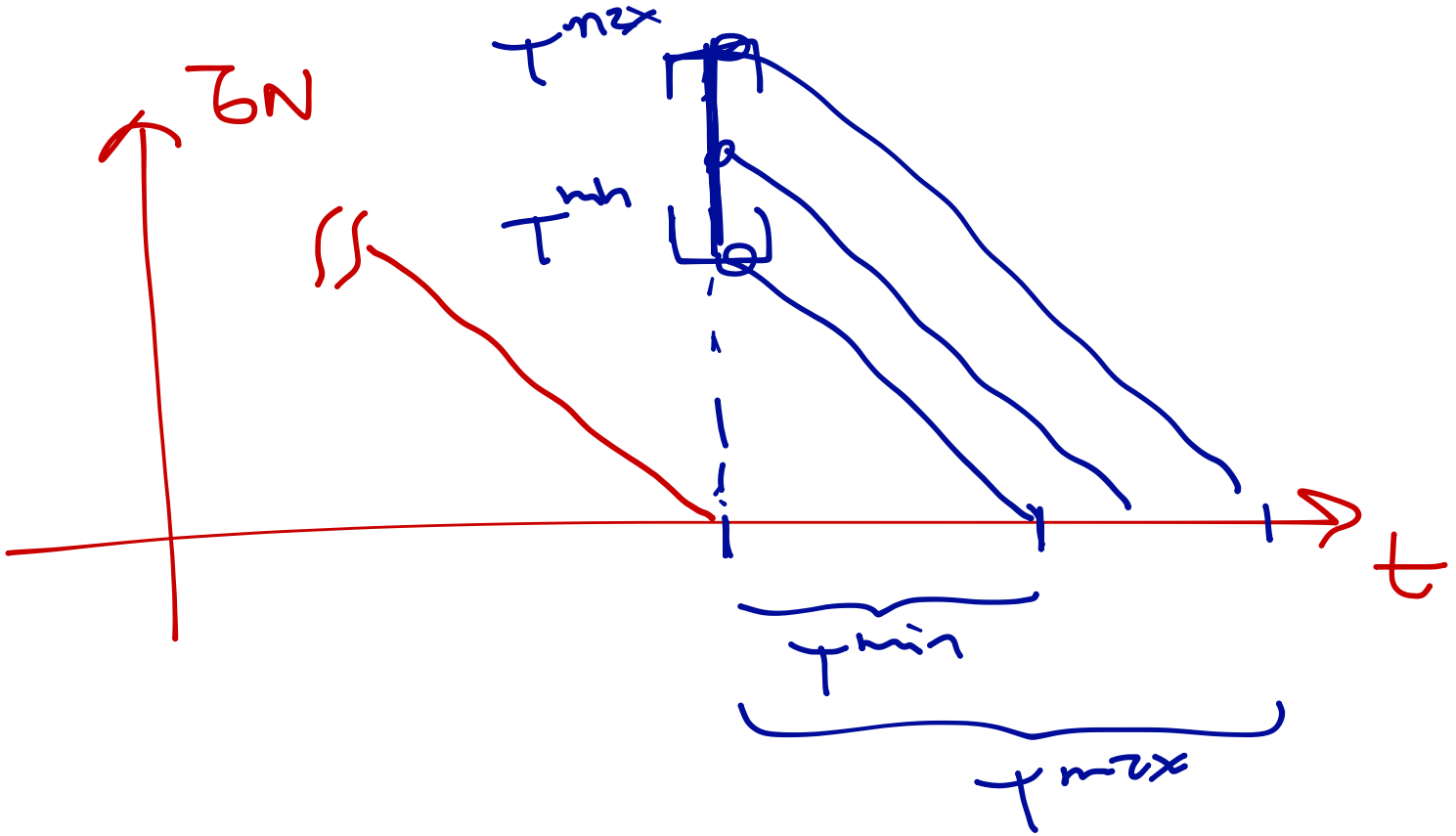
and i^* is the number of transmission events, which might be finite or infinite.

Digital Networks

A mathematical model capturing the said mechanism is given by

$$\dot{\tau}_N = -1, \quad \dot{m}_N = 0 \qquad \text{when} \quad \tau_N \in [0, T_N^{*\max}]$$

$$\tau_N^+ \in [T_N^{*\min}, T_N^{*\max}], \quad m_N^+ = v_N \qquad \text{when} \quad \tau_N \leq 0$$



Models of Systems at The Interface

In general, the interface will be modeled as a hybrid inclusion with state λ, input w, output ψ, and dynamics

$$\begin{aligned} \dot{\lambda} &\in F_I(\lambda, w) && \text{when } (\lambda, w) \in C_I \\ \lambda^+ &\in G_I(\lambda, w) && \text{when } (\lambda, w) \in D_I \\ \psi &= \varphi(\lambda) \end{aligned}$$

where F_I defines the continuous dynamics on C_I and G_I the discrete dynamics on D_I of the interface.

Interconnection Between Physical and Cyber Components

Mathematical models of certain types of cyber-physical systems can be obtained by appropriately interconnecting the models of physical and cyber components.

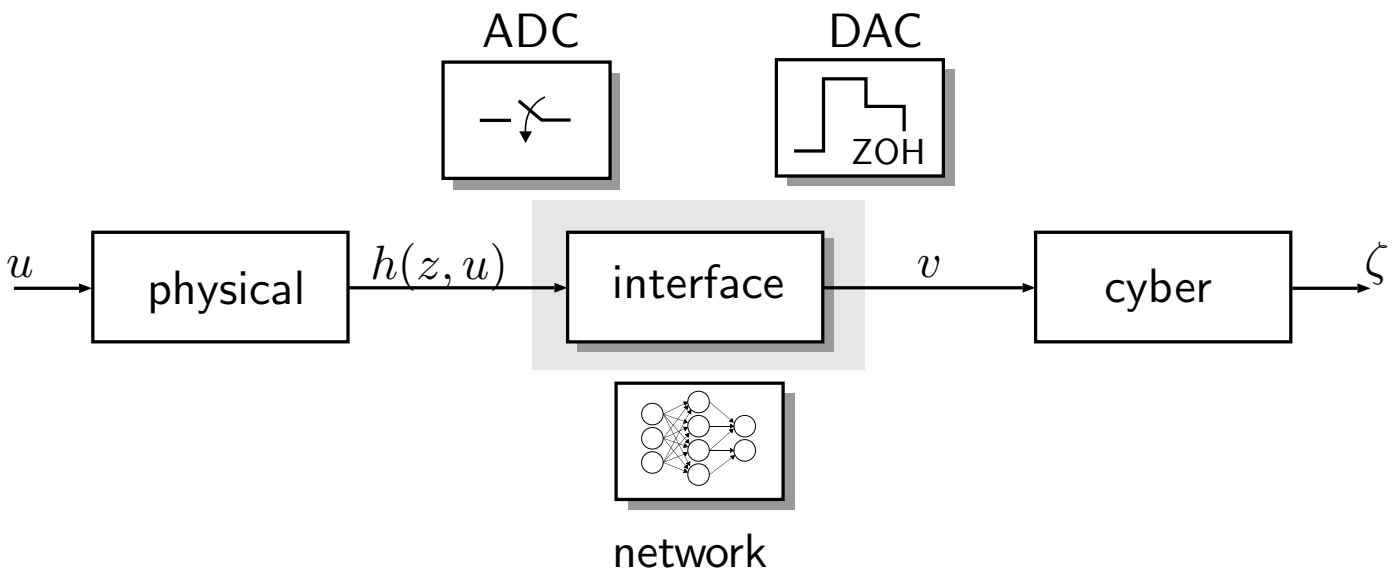


Figure : **Cyber-physical systems:** series interconnections between a plant (part of the physical component), controller (part of the cyber component), and interfaces/converters/signal conditioners (part of both the physical and cyber components).

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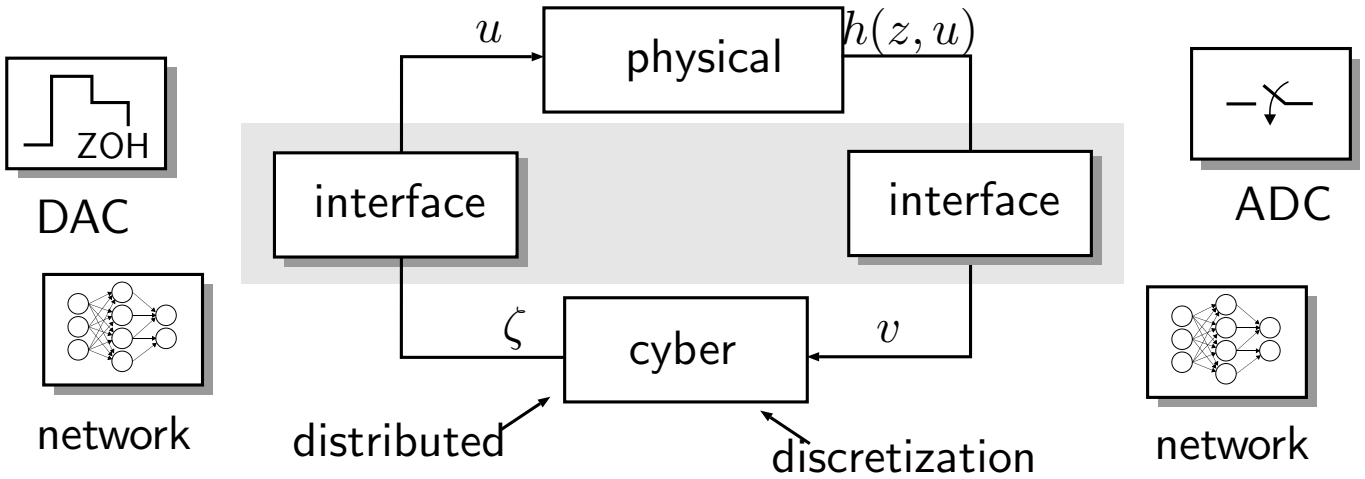

Figure : **Cyber-physical systems:** feedback interconnections between a plant (part of the physical component), controller (part of the cyber component), and interfaces/converters/signal conditioners (part of both the physical and cyber components).

An Full Model of CPSs

A full model of a CPSs in terms of a hybrid inclusion has state $x = (z, \eta, \lambda)$, input $\gamma = (u, v, w)$, and dynamics

$$\dot{x} \in F(x, \gamma) \qquad (x, \gamma) \in C \tag{1}$$

$$x^+ \in G(x, \gamma) \qquad (x, \gamma) \in D \tag{2}$$

- The map F is the *flow map*

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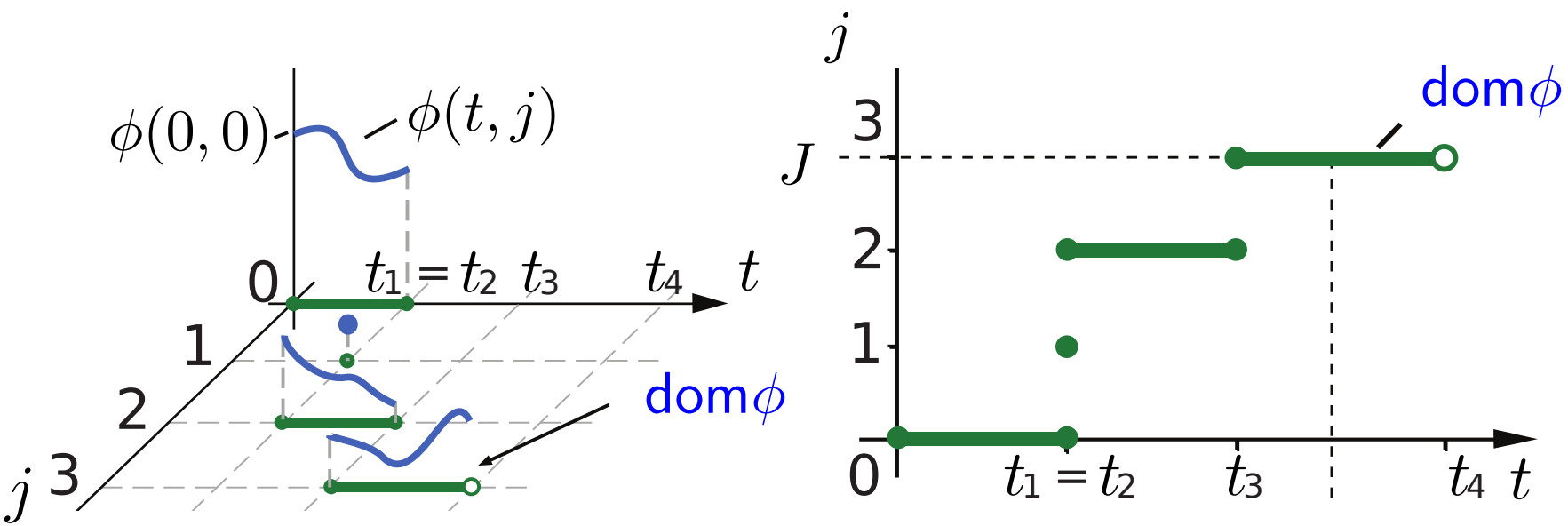
Solutions to CPSs

Hybrid time:

- Flows parameterized by $t \in \mathbb{R}_{\geq 0} := [0, +\infty)$.

A solution ϕ to $\mathcal{H}$ is a function given on hybrid time domains, called hybrid arc:



Solutions to CPSs

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- Jumps parameterized by $j \in \mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\}$.

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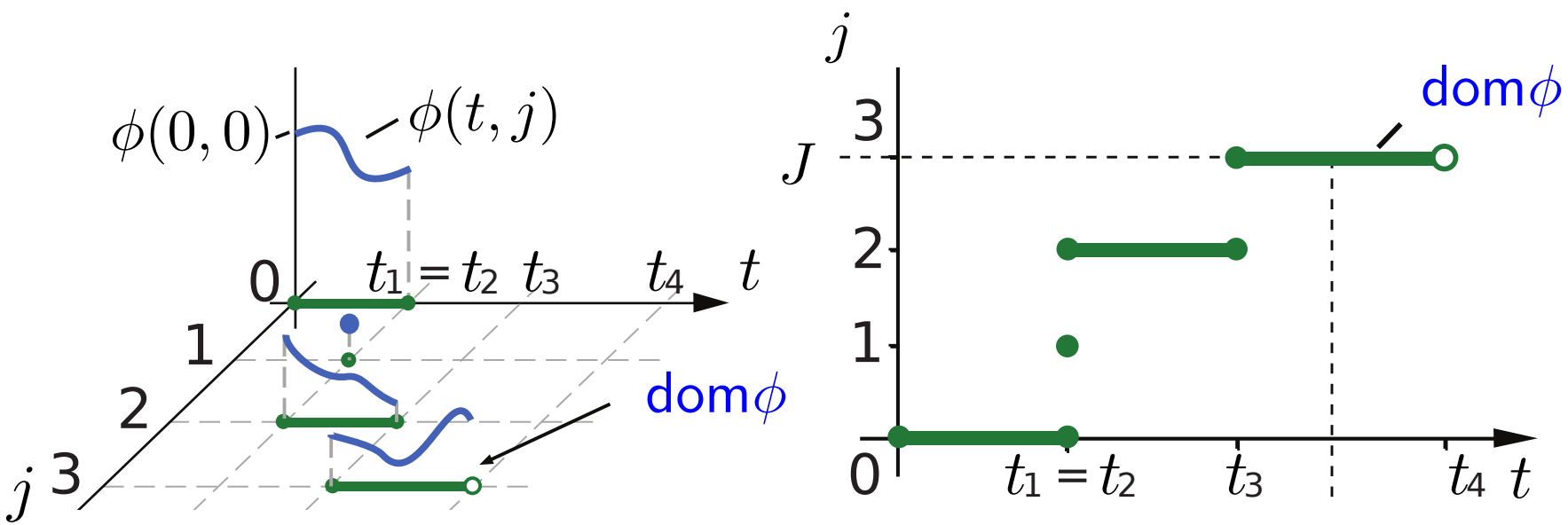

Implementation of a FSM

Transitions of the state of a FSM are triggered by changes of its input v. When the input to the FSM is an external signal, a digital implementation of a FSM would require a conversion of the input, from analog to digital.

- Conversion occurs at isolated time instants t_i

$$\left.\begin{array}{rcl} \dot{\tau}_s & = & 1 \\ \dot{q} & = & 0 \end{array}\right\} \quad \text{when} \quad (q, v) \in Q \times \Sigma, \tau_s \in [0, T_s^*]$$

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Simulating CPSs

The coupling between physics and computations in cyber-physical systems makes their simulation difficult. A simulator for cyber-physical systems has to be capable of computing the solution while evolving according to the physics of the system while monitoring the variables for a potential event triggering a discrete transition in the cyber components.

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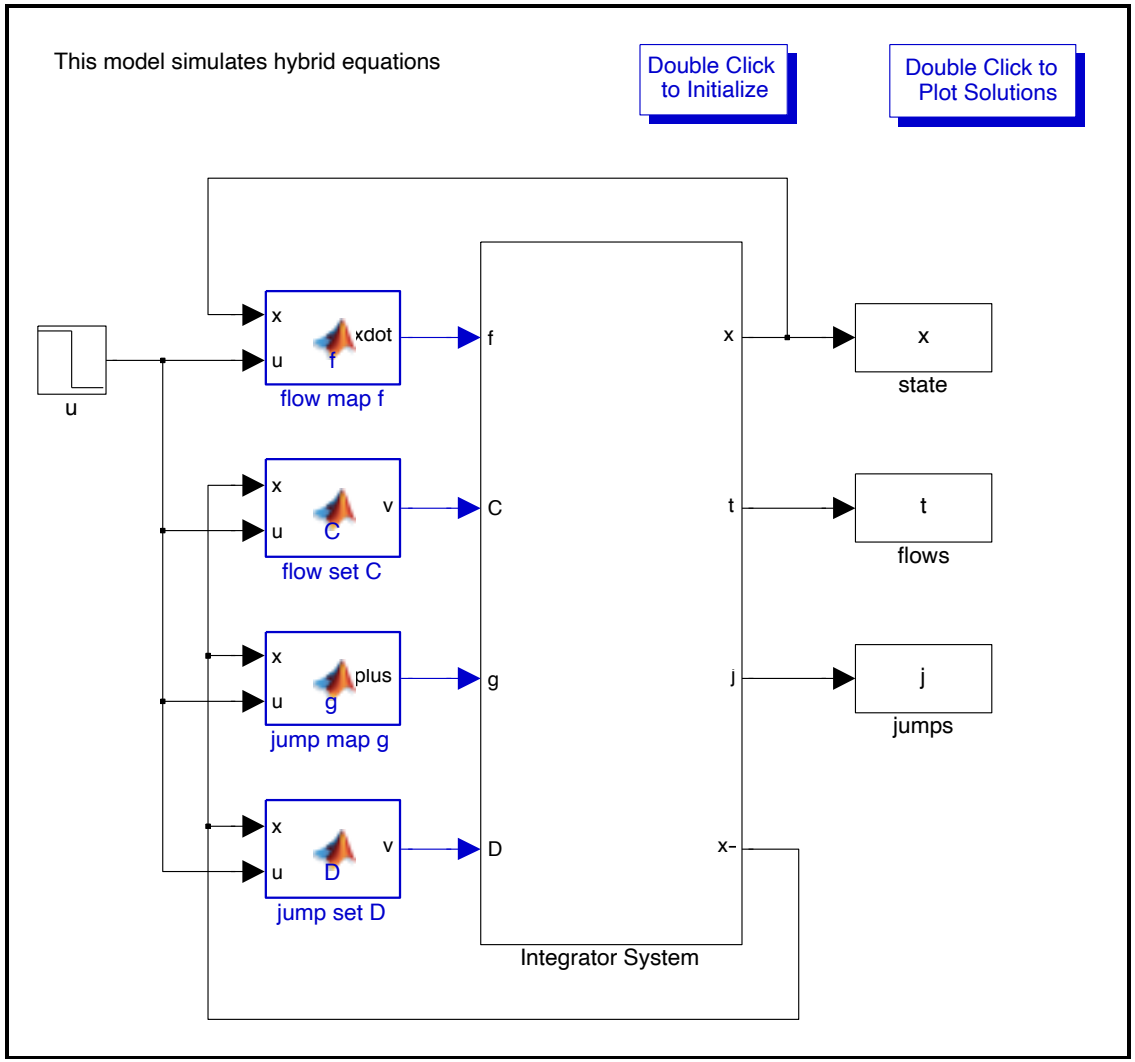
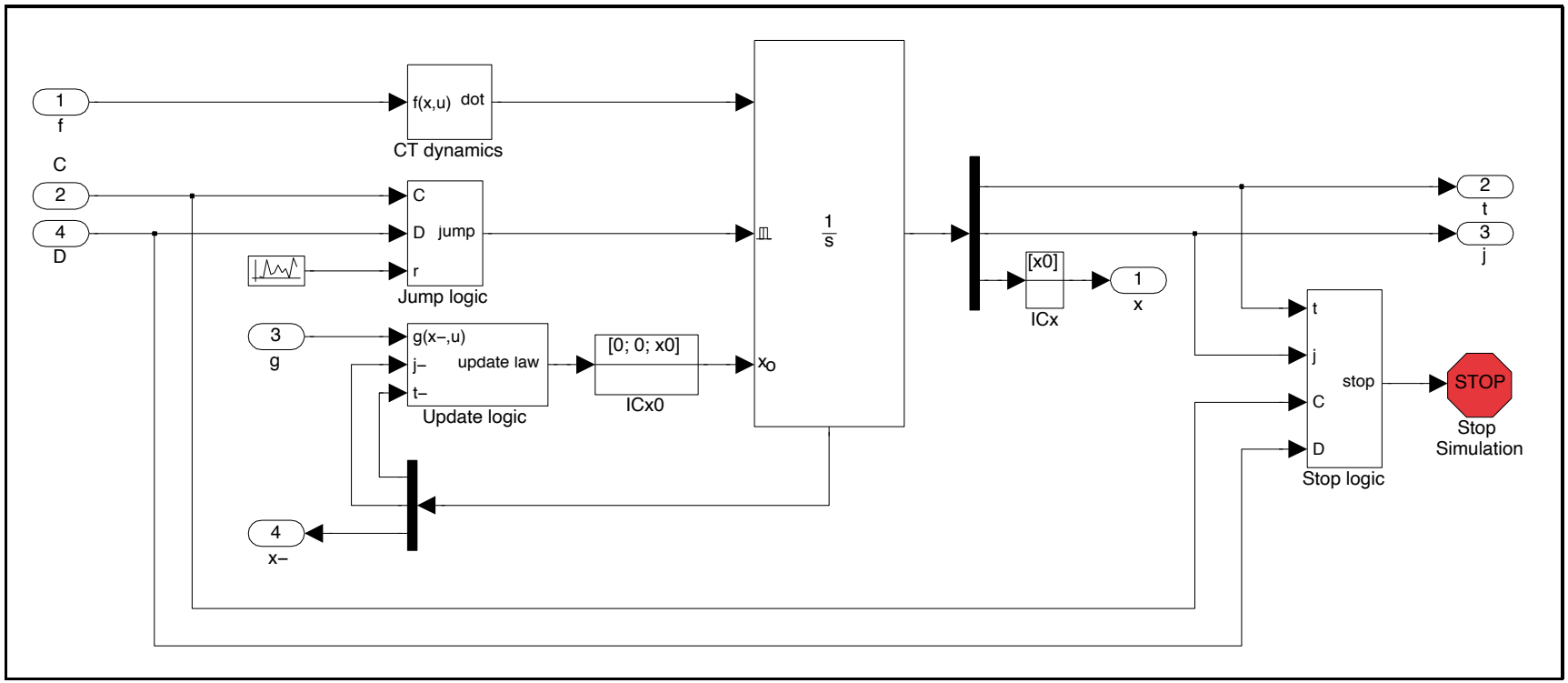
- A recent software package developed for the simulation of hybrid equations within Matlab/Simulink, namely, the Hybrid Equations (HyEQ) Toolbox

Simulating CPSs



Matlab/Simulink implementation of a hybrid inclusion with data (C, f, D, g) with inputs.

Simulating CPSs



Internals of integrator system.

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