Controllability of systems defined on graphs

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joint work with

(in alphabetical order)

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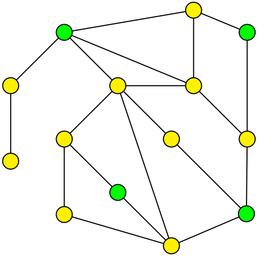
outline

- diffusively coupled leader/follower systems
- controllability (a geometric approach)
- graph partitions
- lower/upper bounds for controllable subspace
- systems defined on graphs
- zero forcing and leader selection
- conclusions

diffusively coupled leader/follower systems



p agents $V = \{1, 2, \ldots, p\}$

communication topology
$G = (V, E)$ an undirected graph

ℓ leaders $V_L = \{v_1, v_2, \ldots, v_\ell\}$

$p - \ell$ followers $V_F = V \setminus V_L$

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followers (yellow node): $\dot{x}_i = z_i$

leaders (green node): $\dot{x}_i = z_i + \boldsymbol{u_k}$ where $i = v_k \in V_L$

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diffusive coupling $z_i = \sum_{(i,j) \in E} (x_j - x_i)$

overall dynamics

if i is a follower $\quad \dot{x}_i = z_i$

if i is the kth leader $\quad \dot{x}_i = z_i + u_k \quad$ where $i = v_k \in V_L$

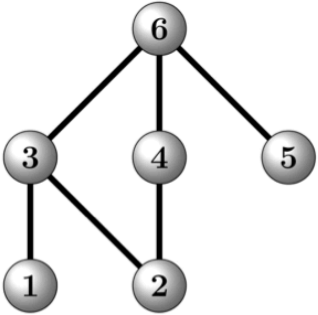
diffusive coupling $\quad z_i = \sum_{(i,j)\in E}(x_j - x_i)$

$$x = \mathrm{col}(x_1, x_2, \ldots, x_p) \qquad u = \mathrm{col}(u_1, u_2, \ldots, u_\ell)$$

$$\dot{x} = -Lx + Mu$$

$$M_{ik} = \begin{cases} 1 & \text{if } i = v_k \\ 0 & \text{otherwise.} \end{cases}$$

Laplacian of a graph



adjacency matrix:

$$A_{ij}(G) = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

degree matrix:

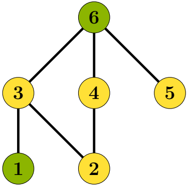
$$D(G) = \operatorname{diag}(d_1, d_2, \dots, d_N)$$

Laplacian:

$$L(G) = D(G) - A(G)$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

example



$$V = \{1, 2, \dots, 6\}$$

$$V_L = \{1, 6\}$$

$$V_F = \{2, 3, 4, 5\}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}}_{L} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{M} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\operatorname{rank} \begin{bmatrix} M & LM & \cdots & L^5 M \end{bmatrix} = 6.$$

controllability of diffusively coupled leader/follower systems

$$\dot{x} = Lx + Mu$$

- graph topological conditions
- leader selection method
- minimum number of leaders

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geometric approach

- A subspace $\mathcal{W} \subseteq \mathbb{R}^p$ is L-invariant if $\quad Lw \in \mathcal{W} \quad$ whenever $\quad w \in \mathcal{W}$.
- A subspace $\mathcal{W} = \operatorname{im} W$ is L-invariant if and only if there exists a matrix Q such that $\quad LW = WQ$. $\qquad \operatorname{im} \equiv$ range
- The controllable subspace

$$\langle L \mid \operatorname{im} M \rangle := \operatorname{im} M + L \operatorname{im} M + \cdots + L^{p-1} \operatorname{im} M = \operatorname{im} \begin{bmatrix} M & LM & \cdots & L^{p-1}M \end{bmatrix}$$

is the smallest L-invariant subspace that contains $\operatorname{im} M$.

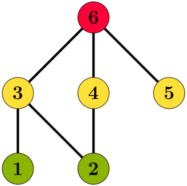
partitions

$G = (V, E)$

Any subset C of V is called a *cell*.

A collection of cells $\pi = \{C_1, C_2, \ldots, C_k\}$ is called a *partition* if $C_i \cap C_j = \varnothing$ for $i \neq j$ and $\cup_i C_i = V$.

The *characteristic matrix* of a partition π is defined as $$P_{ij}(\pi) = \begin{cases} 1 & \text{if } i \in C_j \\ 0 & \text{otherwise.} \end{cases}$$



$$\pi = \big\{\{1,2\},\{3,4,5\},\{6\}\big\} \qquad P(\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

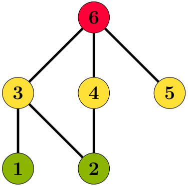
distance partition

Def. For any two vertices $v, w \in V$, $\mathrm{dist}(v,w)$ = the length of the shortest path.

Def. For a given vertex $v \in V$, let $C_i = \{w \in V \mid \mathrm{dist}(v,w) = i\}$ for $i \geqslant 1$. The *distance partition w.r.t.* v is the partition

$$\pi_D(v) = \{\{v\}, C_1, C_2, \ldots, C_k\}.$$

Ex.



$v = 6$

$C_1 = \{3, 4, 5\}$

$C_2 = \{1, 2\}$

$\pi_D(6) = \{\{6\}, \{3,4,5\}, \{1,2\}\}$

a lower bound for the controllable subspace

Thm. Consider a multi-agent system $\dot{x} = Lx + Mu$ with the leader set V_L. Then,

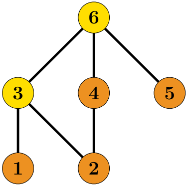
$$\dim\big(\langle L \mid \operatorname{im} M\rangle\big) \geqslant \max\{\operatorname{card}(\pi_D(v)) \mid v \in V_L\}.$$

Question. Upper bound?

almost equitable partitions

Def. A partition $\pi = \{C_1, C_2, \ldots, C_k\}$ is said to be *almost equitable* if for any pair (i, j) with $1 \leqslant i \neq j \leqslant k$, there exists a number b_{ij} such that any vertex $v \in C_i$ has b_{ij} neighbors in C_j.

Ex. $\pi = \big\{\{1, 2, 4, 5\}, \{3, 6\}\big\}$



$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{P(\pi)} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{P(\pi)} \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

almost equitability and L-invariance

Lem. A partition π is almost equitable if and only if $\operatorname{im} P(\pi)$ is L-invariant.

Fact. The controllable subspace $\langle L \mid \operatorname{im} M \rangle$ is the smallest L-invariant subspace containing $\operatorname{im} M$.

Obs. Consider a multi-agent system $\dot{x} = Lx + Mu$ with the leader set $V_L = \{v_1, v_2, \ldots, v_\ell\}$. If

$$\pi = \big\{\{v_1\}, \{v_2\}, \ldots, \{v_\ell\}, C_1, C_2, \ldots, C_k\big\}$$

is an almost equitable partition, then

$\operatorname{im} P(\pi)$ is L-invariant and $\operatorname{im} M \subseteq \operatorname{im} P(\pi)$.

Hence,

$$\langle L \mid \operatorname{im} M \rangle \subseteq \operatorname{im} P(\pi).$$

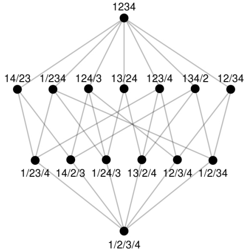
a partial order on partitions

Def. A partition π_1 is *finer* than π_2 if each cell of π_1 is a subset of some cell of π_2.

Notation. $\pi_1 \leqslant \pi_2$ **Ex.** $\big\{\{1,2\},\{3,4\},\{5\},\{6\}\big\} \leqslant \big\{\{1,2\},\{3,4,5\},\{6\}\big\}$.

Obs. $\pi_1 \leqslant \pi_2 \iff \operatorname{im} P(\pi_1) \supseteq \operatorname{im} P(\pi_2)$.

Fact. The set of all partitions Π is a complete lattice w.r.t. "$\leqslant$".



Bell numbers

p	B_p
1	1
2	2
3	5
4	15
5	52
6	203

maximal almost equitable partition

Lem. Let Π_{AEP} denote the set of all almost equitable partitions. For a given partition π_0, define

$$\Pi_{AEP}(\pi_0) = \{\pi \mid \pi \in \Pi_{AEP} \text{ and } \pi \leqslant \pi_0\}.$$

Then,

$$\pi^*_{AEP}(\pi_0) := \sup \Pi_{AEP}(\pi_0)$$

is almost equitable.

controllability

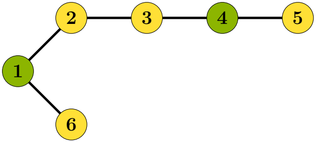
Thm. Consider a multi-agent system with single integrator dynamics $\dot{x} = Lx + Mu$ with the leader set $V_L = \{v_1, v_2, \ldots, v_\ell\}$. Let

$$\pi_L = \big\{\{v_1\}, \{v_2\}, \ldots, \{v_\ell\}, V \setminus V_L\big\}.$$

Then,

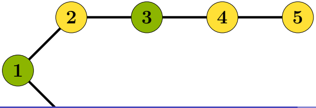
$$\max\{\mathrm{card}(\pi_D(v)) \mid v \in V_L\} \leqslant \dim\big(\langle L \mid \mathrm{im}\, M\rangle\big)$$

$$\langle L \mid \mathrm{im}\, M\rangle \subseteq \mathrm{im}\, P(\pi^*_{AEP}(\pi_L)).$$



lower bound $=$ $\dim\big(\langle L \mid \mathrm{im}\, M\rangle\big)$

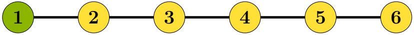
$\langle L \mid \mathrm{im}\, M\rangle \subset$ upper bound.



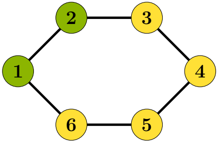
lower bound $<$ $\dim\big(\langle L \mid \mathrm{im}\, M\rangle\big)$

leader selection

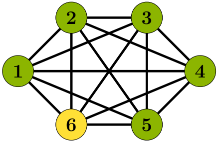
- path graphs (minimum $= 1$)



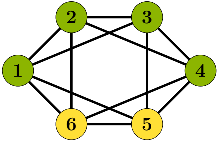
- cycle graphs (minimum $= 2$)



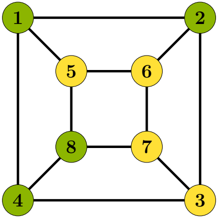
- complete graphs (minimum $= p - 1$)



- circulant graphs (not minimum)



- distance-regular graphs (not minimum)



diffusively coupled leader/follower systems

- p agents $V = \{1, 2, \ldots, p\}$
- communication topology $G = (V, E)$ an undirected graph
- ℓ leaders $V_L = \{v_1, v_2, \ldots, v_\ell\}$ with the dynamics

$$\dot{x}_i = Ax_i + Cz_i + \boldsymbol{Bu_k} \qquad \text{where } i = v_k \in V_L$$

- $p - \ell$ followers $V_F = V \setminus V_L$ with the dynamics

$$\dot{x}_i = Ax_i + Cz_i \qquad \text{where } i \in V_F$$

- $x_i \in \mathbb{R}^n$ state $\quad z_i \in \mathbb{R}^q$ coupling $\quad u_j \in \mathbb{R}^m$ input
- diffusive coupling

$$z_i = K \sum_{(i,j) \in E} (x_j - x_i)$$

overall dynamics

if i is the kth leader $\quad \dot{x}_i = Ax_i + Cz_i + Bu_k$

if i is a follower $\quad \dot{x}_i = Ax_i + Cz_i$

diffusive coupling $\quad z_i = K \sum_{(i,j)\in E}(x_j - x_i)$

$$x = \mathrm{col}(x_1, x_2, \ldots, x_p) \qquad u = \mathrm{col}(u_1, u_2, \ldots, u_\ell)$$

$$\dot{x} = \overbrace{(I \otimes A - L \otimes CK)}^{\hat{L}}\, x + \overbrace{(M \otimes B)}^{\hat{M}}\, u$$

$$M_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases}$$

controllability of diffusively coupled leader/follower systems

Thm. The system $\dot{x} = \hat{L}x + \hat{M}u$ where

$$\hat{L} = I \otimes A - L \otimes CK \quad \text{and} \quad \hat{M} = M \otimes B$$

is controllable if and only if

1. $\Sigma(L, M)$ is controllable, and
2. $\Sigma(A - \lambda CK, B)$ is controllable for each $\lambda \in \sigma(L)$.

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second condition is automatically satisfied by

- single integrator (obvious)
- double integrator (Goldin and Raisch, 2010)
- higher order integrators (Jiang et al, 2009)

systems defined on graphs

so far $\dot{x} = Lx + Mu$

L is the Laplacian of a graph $G = (V, E)$ and M encodes a leader set $V_L \subseteq V$.

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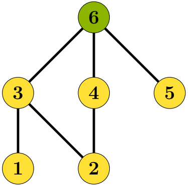
from now on $\dot{x} = Xx + Mu$

where $X \in \mathcal{Q}(G)$ with $\mathcal{Q}(G)$ is the *qualitative class* of a graph $G = (V, E)$

$$\mathcal{Q}(G) = \{Y \mid Y_{ij} \neq 0 \text{ for } i \neq j \Leftrightarrow (i,j) \in E\}$$

and M encodes a leader set $V_L \subseteq V$.



$$X = \begin{bmatrix} ? & 0 & * & 0 & 0 & 0 \\ 0 & ? & * & * & 0 & 0 \\ * & * & ? & 0 & 0 & * \\ 0 & * & 0 & ? & 0 & * \\ 0 & 0 & 0 & 0 & ? & * \\ 0 & 0 & * & * & * & ? \end{bmatrix} \qquad M = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$*$: nonzero $\quad ?$: arbitrary

- $(X; V_L)$ is controllable $:\Longleftrightarrow$ (X, M) is controllable.

zero forcing

recipe given a graph $G = (V, E)$ and a subset W of V

1. color the vertices in W black
2. if $\mathbf{w}$ is a black and has only one white neighbor $\mathbf{v}$, color $\mathbf{v}$ black
3. repeat the step 2 until there is no white to color black
4. define $D(W)$, the derived set of W, as the set of all black vertices at the end

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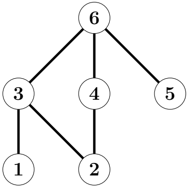
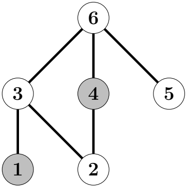
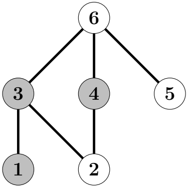
Ex.





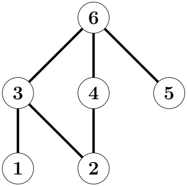
$W = \{1, 4\}$

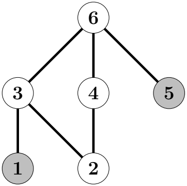


$D(W) = \{1, 3, 4\}$

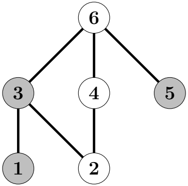
zero forcing cont'd

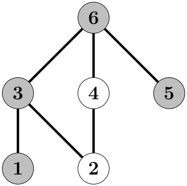
Another ex.

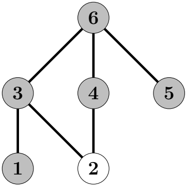


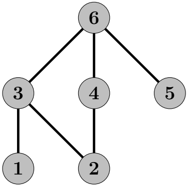


$W = \{1, 5\}$









$D(W) = \{1, 2, \ldots, 6\}$

zero forcing and controllability

$G = (V, E)$

$\dot{x} = Xx + Mu$ where $X \in \mathcal{Q}(G) := \{Y \mid Y_{ij} \neq 0 \text{ for } i \neq j \Leftrightarrow (i,j) \in E\}$ and M encodes a leader set $V_L \subseteq V$.

$(G; V_L)$ is controllable $:\Longleftrightarrow$ $(X; V_L)$ is controllable for all $X \in \mathcal{Q}(G)$.

Thm. (G, V_L) is controllable $\Longleftrightarrow$ $(G, D(V_L))$ is controllable.

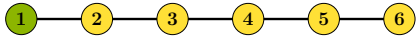
Def. A set $W \subseteq V$ is a *zero forcing set* for G if $D(W) = V$. The *zero forcing number* $Z(G)$ is the minimum of $\mathrm{card}(W)$ over all zero forcing sets W.

Thm. The following statements hold:

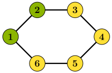
1. (G, V_L) is controllable $\Longleftrightarrow$ V_L is a zero forcing set.
2. $\ell_{\min}(G) = Z(G)$.

zero forcing set and leader selection

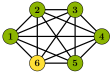
- in general computing the zero forcing number is an NP-hard problem (Aazami, 2008)
- for tree graphs: polynomial-time algorithms exist (Hogben, 2010)
- existing leader selection methods for Laplacian controllability are actually based on zero forcing set/number:
  - path graphs (minimum $= 1$)



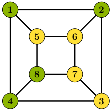
  - cycle graphs (minimum $= 2$)



  - complete graphs (minimum $= p - 1$)



  - distance-regular graphs (not necessarily minimum)



conclusions

controllability for

- $\dot{x} = Lx + Mu$: graph topological lower/upper bounds
- $\dot{x} = (I \otimes A - L \otimes CK)x + (M \otimes B)u$: algebraic necessary and sufficient conditions
- $\dot{x} = Xx + Mu$, $X \in \mathcal{Q}(G)$: topological necessary and sufficient conditions and leader selection method

extensions:

- all results are still valid if the graph $G = (V, E)$ directed and/or weighted
- stronger results for special classes of graphs (such as path, cycle, complete, tree, distance-regular, etc.)
- output controllability
- controllability of switching systems defined on graphs

future work:

- applications of zero forcing notions to other control problems
  - disturbance decoupling problem
  - fault detection