Model reduction of multi-agent systems

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outline

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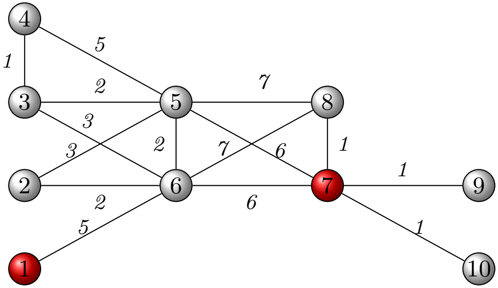
introduction



- General problem: given a large scale networked multi-agent system with leaders and followers, approximate it by a lower order network, while preserving (some of) the network structure.
- Does the approximated network preserve important properties such as *consensus*?
- How good is the approximation? Can one compute (upper and lower bounds on) the error?

graph theory

- Weighted directed graph $G = (V, E, A)$, $V = \{1, 2, \dots, p\}$, E set of arcs, adjacency matrix $A = [a_{ij}]$.
- Laplacian of G: $L = D - A$, $D = \mathrm{diag}(d_1, d_2, \dots, d_p)$, $d_i = \sum_j a_{ij}$.
- Incidence matrix of G: $R = [r_{ij}]$ with

$$r_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is the head of arc } j \\ -1 & \text{if vertex } i \text{ is the tail of arc } j \\ 0 & \text{otherwise} \end{cases}$$

  for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$ (q is the total number of arcs).
- Incidence matrix *undirected* graph: assign arbitrary orientation to the edges and take the incidence matrix of the corresponding directed graph.
- Edge weight matrix $W = \mathrm{diag}(w_1, w_2, \dots, w_q)$ with w_j is the weight associated to the edge (arc) j, $j = 1, 2, \dots, q$.
- For undirected graphs Laplacian: $L = RWR^T$.

leader follower multi-agent systems

$G = (V, E, A)$ weighted undirected graph

leaders $V_L = \{v_1, v_2, \ldots, v_\ell\} \subseteq V$ leaders, followers $V_F := V \backslash V_L$

Leader-follower multi-agent system:

$$\dot{x}_i = \begin{cases} \sum_{j=1}^{p} a_{ij}(x_j - x_i) & \text{if } i \in V_F \\ \sum_{j=1}^{p} a_{ij}(x_j - x_i) + u_k & \text{if } i \in V_L \end{cases}$$

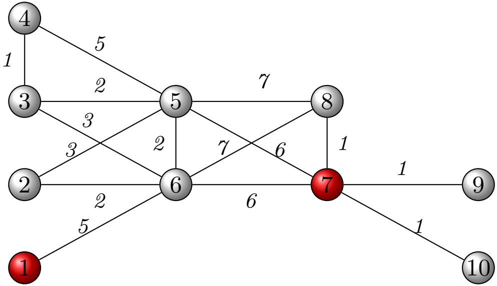
$x_i \in \mathbb{R}$ state of agent i, $u_k \in \mathbb{R}$ external input applied to agent $i = v_k$.
In compact form

$$\dot{x} = -Lx + Mu,$$

with L the Laplacian of G, M is given by

$$M_{ik} = \begin{cases} 1 & \text{if } i = v_k \\ 0 & \text{otherwise.} \end{cases}$$

example



$$\dot{x} = -Lx + Mu$$

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 6 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 & 25 & -2 & -6 & -7 & 0 & 0 \\ -5 & -2 & -3 & 0 & -2 & 25 & -6 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & -6 & 15 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -7 & -7 & -1 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

problem statement

Problem

Approximate the leader-follower multi-agent system

$$\dot{x} = -Lx + Mu$$

by a reduced order multi-agent system

$$\dot{\hat{x}} = -\widehat{L}\hat{x} + \widehat{M}u$$

such that the spatial structure of the network is preserved.

Idea

We want to do this by reducing the size of the network graph through clustering of the agents/vertices

Petrov-Galerkin projections

Consider an arbitrary system

$$\dot{x} = Ax + Bu \qquad y = Cx$$

with state space $\mathbb{R}^n$. Let $W, V \in \mathbb{R}^{n \times r}$ (with $r < n$) such that $W^T V = I_r$. Then VW^T is a projection.

A reduced order model (projected model) is then given by

$$\dot{\widehat{x}} = W^T AV\widehat{x} + W^T Bu \qquad y = CV\widehat{x}$$

with reduced order state space $\mathbb{R}^r$.

General reduction framework: Krylov based, truncation methods, moment matching use this projection with appropriate choice of matrices V and W.

Idea

Put idea of clustering of network graph into the Petrov-Galerkin framework

graph partitions

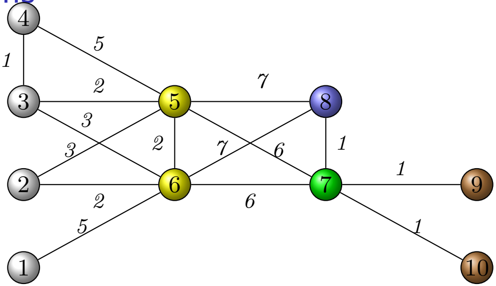
- $V = \{1, 2, \ldots, p\}$ vertex set of a graph G.
- Any nonempty subset of V is called a cell of V.
- Collection of cells, given by $\pi = \{C_1, C_2, \ldots, C_r\}$, called a partition of V if $\cup_i C_i = V$ and $C_i \bigcap C_j = \varnothing$ whenever $i \neq j$.
- For a cell $C \subset V$, the characteristic vector of C is the p-dimensional column vector $p(C)$ with

$$p_i(C) = \begin{cases} 1 & \text{if } i \in C, \\ 0 & \text{otherwise.} \end{cases}$$

- For a partition $\pi = \{C_1, C_2, \ldots, C_r\}$, the characteristic matrix of π is

$$P(\pi) = \begin{bmatrix} p(C_1) & p(C_2) & \cdots & p(C_r) \end{bmatrix}.$$

graph partitions



- Vertex set: $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
- Cell: any nonempty subset of V
- Partition: $\pi = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\}$
- Characteristic matrix:

$$P(\pi) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T .$$

projection by graph partitions

- A direct application of Petrov-Galerkin projections will destroy the spatial structure of the network (e.g. balanced truncation, moment matching)
- Instead: choose a graph partition, characteristic matrix $P(\pi)$.

$$W := P(\pi)(P^T(\pi)P(\pi))^{-1}$$

$$V := P(\pi).$$

- Corresponding reduced order system:

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \widehat{M}u,$$

  - $\hat{L} = W^T L V = (P^T P)^{-1} P^T L P$
  - $\widehat{M} = W^T M = (P^T P)^{-1} P^T M$

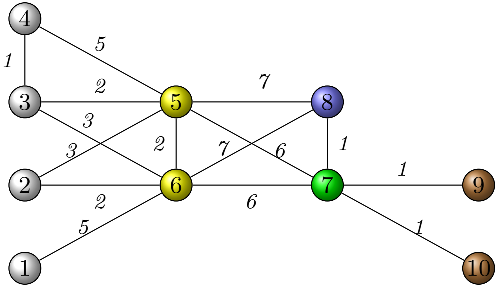
properties of partition-based projection

- $\widehat{L}$ is the Laplacian of a weighted *directed* graph, $\widehat{G} = (\widehat{V}, \widehat{E}, \widehat{A})$.
- Each cell of π in G becomes a vertex in $\widehat{G}$.
- There is an arc from vertex α to vertex β in $\widehat{G}$ if and only if there exist $i \in C_\alpha$ and $j \in C_\beta$ with $\alpha \neq \beta$ such that $\{i, j\} \in E$.
- Therefore, $\widehat{G}$ is a *symmetric directed* graph, i.e. $(\alpha, \beta) \in \widehat{E} \Leftrightarrow (\beta, \alpha) \in \widehat{E}$.
- Relationship between A and $\widehat{A} = [\widehat{a}_{\alpha\beta}]$ given by

$$\widehat{a}_{\alpha\beta} = \frac{1}{|C_\alpha|} \sum_{i \in C_\alpha, j \in C_\beta} a_{ij},$$

  for $\alpha \neq \beta$.
- The input weights in $\widehat{M}$ depend on the cardinality of the leader cells.

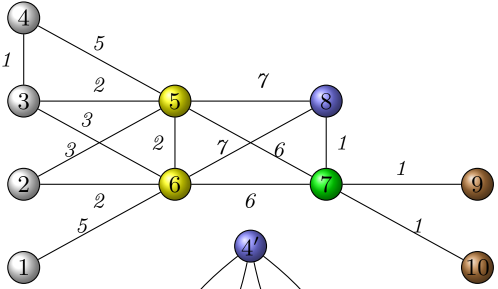
relationship between the original and the reduced model



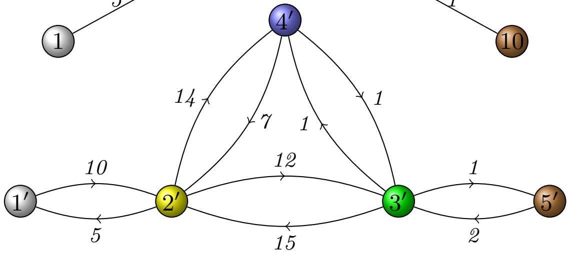
- The reduced order model: $\dot{\hat{x}} = -\widehat{L}\hat{x} + \widehat{M}u$
  - $$\widehat{L} = \begin{bmatrix} 5 & -5 & 0 & 0 & 0 \\ -10 & 23 & -6 & -7 & 0 \\ 0 & -12 & 15 & -1 & -2 \\ 0 & -14 & -1 & 15 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \qquad \widehat{M} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

relationship between the original and the reduced model

- Original graph:



- Reduced graph:



preservation of consensus

- If the weighted undirected graph G is connected, then the multi-agent system $\dot{x} = -Lx$ reaches *consensus*, i.e. for all $i, j \in V$ we have $x_i(t) - x_j(t) \to 0$ as $t \to \infty$.
- Since $\widehat{L} = (P^T P)^{-1} P^T L P$, we see that

$$(P^T P)^{\frac{1}{2}} \widehat{L} (P^T P)^{-\frac{1}{2}} = (P^T P)^{-\frac{1}{2}} P^T L P (P^T P)^{-\frac{1}{2}}.$$

  Thus $\widehat{L}$ is similar to a symmetric matrix, and hence has *only real eigenvalues*.
- Moreover: the eigenvalues $\widehat{\lambda}_i$ of $\widehat{L}$ *interlace* the eigenvalues λ_i of L:

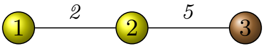
$$\lambda_i \leqslant \widehat{\lambda}_i \leqslant \lambda_{p-r+i}$$

  for $i = 1, 2 \ldots, r$.
- In particular $\lambda_2 \leqslant \widehat{\lambda}_2$ so the reduced order system $\dot{\widehat{x}} = -\widehat{L}\widehat{x}$ reaches consensus with rate of convergence at least as fast as the original network.

input-output approximation of networks

- $G = (V, E, A)$ undirected weighted graph with incidence matrix R, and edge weight matrix W.
- Original model: $\dot{x} = -Lx + Mu$
- We add output variables as: $y = W^{\frac{1}{2}} R^T x$. Note that $\|y\|^2 = x^T L x$.
- Example:

$$y = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



- Output variables after projection: $\widehat{y} = W^{\frac{1}{2}} R^T P \widehat{x}$
- In this example: $$\widehat{y} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix}$$

model reduction error

Original model:

$$\dot{x} = -Lx + Mu$$
$$y = W^{\frac{1}{2}} R^T x$$

Reduced model:

$$\dot{\widehat{x}} = -\widehat{L}\widehat{x} + \widehat{M}u$$
$$\widehat{y} = W^{\frac{1}{2}} \widehat{R}^T \widehat{x}$$

- Transfer matrices

$$S(s) = W^{\frac{1}{2}} R^T (sI + L)^{-1} M \qquad \text{and} \qquad \widehat{S}(s) = W^{\frac{1}{2}} \widehat{R}^T (sI + \widehat{L})^{-1} \widehat{M}$$

- Note S and $\widehat{S}$ are in $\mathcal{H}_2$. Approximation error (squared $\mathcal{H}_2$-norm):

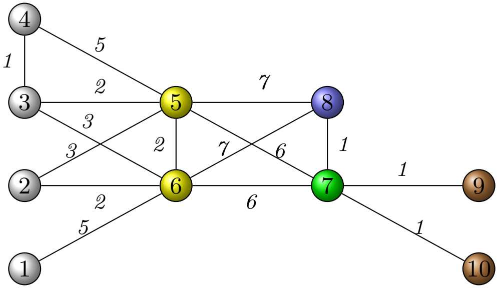
$$\|S - \widehat{S}\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|S(i\omega) - \widehat{S}(i\omega)\|^2 d\omega$$

- Problem: for a given partition, find an expression for this error. Find a priori upper or lower bounds.

almost equitable partitions

Def. For an unweighted undirected graph, a partition $\pi = \{C_1, C_2, \ldots, C_r\}$ is said to be *almost equitable* if for any pair (i, j) with $1 \leqslant i \neq j \leqslant r$, there exists an integer b_{ij} such that any vertex $v \in C_i$ has b_{ij} neighbors in C_j.

Def. Let $G = (V, E, A)$ be a weighted undirected graph. A partition $\pi = \{C_1, C_2, \ldots, C_r\}$ is called *almost equitable* if for any pair (i, j) with $1 \leqslant i \neq j \leqslant r$, there exists a real number b_{ij} such that $\sum_{\substack{(v,w)\in E \\ w\in C_j}} a_{ij} = b_{ij}$ for all $v \in C_i$.



model reduction error

Thm. Let

$\pi = \{C_1, C_2, \ldots, C_r\}$ be an AEP of G,

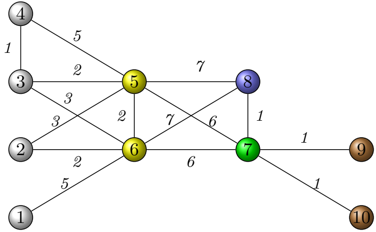
$V_L = \{v_1, v_2, \ldots, v_\ell\}$ leader vertices,

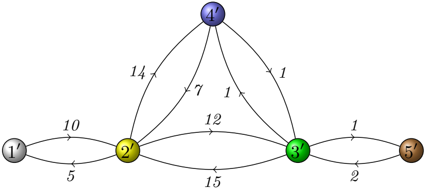
k_i be s.t. $v_i \in C_{k_i} \quad (i = 1, 2 \ldots, \ell)$.

The (normalized) model reduction error is given by

$$E(\pi) = \frac{\|S - \widehat{S}\|_2^2}{\|S\|_2^2} = \frac{\sum_{i=1}^{\ell}(1 - \frac{1}{|C_{k_i}|})}{\ell(1 - \frac{1}{p})}.$$

example





$$V_L = \{1, 7\}$$

$$E(\pi) = \frac{\sum_{i=1}^{\ell}(1 - \frac{1}{|C_{k_i}|})}{\ell(1 - \frac{1}{p})} = \frac{(1 - \frac{1}{4}) + (1 - \frac{1}{1})}{2(1 - \frac{1}{10})} = \frac{5}{12}$$

.

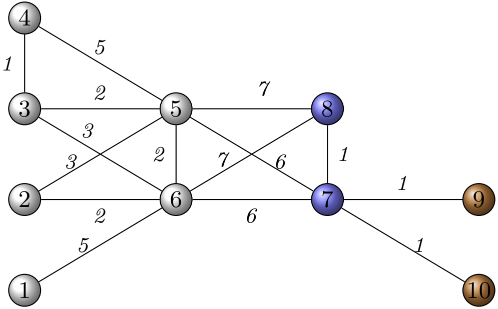
remarks

- $0 \leqslant E(\pi) \leqslant 1$
- $E(\pi) = 0$ for $\pi = \{\{1\}, \{2\}, \dots, \{p\}\}$
- $E(\pi) = 1$ for $\pi = \{V\}$
- $E(\pi)$ determined by the cardinalities of those cells in π that contain the leaders.

Cor. If π is an almost equitable partition with the property that $\{v_i\} \in \pi$ for all $i = 1, 2 \dots, \ell$ (i.e. each leader is the unique element in its cell) then $E(\pi) = 0$.

arbitrary partitions

- What if we start with an *arbitrary* (not necessarily AEP) partition? In that case the closed form expression fails. But: can we make an estimate of $E(\pi)$?



- $\pi = \{\{1,2,3,4,5,6\},\{7,8\},\{9,10\}\}$ is not an an almost equitable partition.

partial ordering of partitions

- Let $G = (V, E, A)$ be a weighted undirected graph. Partial ordering on all partitions of G. Let π_1 and π_2 be two partitions. We say $\pi_1 \leqslant \pi_2$ if every cell in π_1 is a subset of a cell in π_2: π_1 is *finer* then π_2, π_2 is *coarser* than π_1.
- In terms of the characteristic matrices: $\pi_1 \leqslant \pi_2$ if and only if $\operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi_1)$.
- Using the fact that for any AEP π_0, $\operatorname{im} P(\pi_0)$ is L-invariant we obtain:

Thm. Let π_0 be an AEP of G. Then for every partition π that is coarser than π_0 we have $E(\pi_0) \leqslant E(\pi)$.

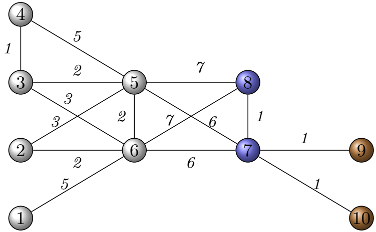
$\pi^*_{\mathrm{AEP}}(\pi)$: *the maximal almost equitable partition finer than* π.

Cor. For any partition π, we have $E(\pi) \geqslant E\big(\pi^*_{\mathrm{AEP}}(\pi)\big)$.

example

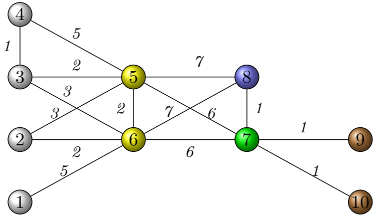
$V_L\{1,7\}$

π



$E(\pi) \geqslant \frac{5}{12}$

$\pi^*_{\mathrm{AEP}}(\pi)$



$E(\pi^*_{\mathrm{AEP}}(\pi)) = \frac{5}{12}$

model reduction by edge removal

so far clustering of the nodes

$$G = (V, E, A) \xrightarrow{P(\pi)} \widehat{G} = (\widehat{V}, \widehat{E}, \widehat{A})$$

undirected weighted → directed symmetric weighted

$$\dot{x} = -Lx + Mu \qquad \dot{\widehat{x}} = -\widehat{L}\widehat{x} + \widehat{M}u$$

$$y = W^{\frac{1}{2}} R^T x \qquad \widehat{y} = W^{\frac{1}{2}} \widehat{R}^T \widehat{x}$$

$$\mathrm{card}(V) > \mathrm{card}(\widehat{V})$$

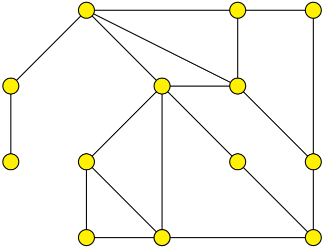
from now on removing edges

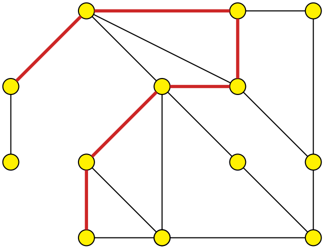
$$G = (V, E) \xrightarrow{\text{edge removal}} \widehat{G} = (V, \widehat{E})$$

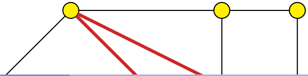
undirected weighted → undirected weighted

$$\mathrm{card}(E) > \mathrm{card}(\widehat{E})$$

paths, cycles, trees







$G = (V, E)$ undirected unweighted.

A path is a sequence of nodes such that sequential nodes are connected.

A cycle is a path where the first and last node in the path are the same.

error bound for reduction by edge removal

Thm. Let $G = (V, E)$ be an undirect unweighted graph with the Laplacian L and incidence matrix R. Consider the system Σ

$$\dot{x} = -Lx + d \qquad y = R^T x.$$

Let T be a spanning tree of G with missing independent cycles $c_1, c_2, \ldots, c_k$. Consider the system $\widehat{\Sigma}$

$$\dot{\widehat{x}} = -\widehat{L}\widehat{x} + d \qquad \widehat{y} = \widehat{R}^T \widehat{x}$$

where $\widehat{L}$ is the Laplacian and $\widehat{R}$ is the incidence matrix of T. Then, the approximation error is given by

$$\|\Sigma - \widehat{\Sigma}\|_{\mathcal{H}_2}^2 \leqslant \frac{1}{2} \sum_{i=1}^{k} \left(\frac{\ell(c_i) - 1}{1 + \frac{\lambda_2}{\lambda_2 + \lambda_p}\big(\ell(c_i) - 1\big)} + \frac{1}{\ell(c_i)} - 1 \right)$$

where λ_2 and λ_p are the smallest nonzero and the largest eigenvalues of $\widehat{L}$.

directions for future research

- Extension to directed weighted graphs.
- Instead of a lower bound on the error associated with an arbitrary partition we would like to find *upper bounds*.
- Extension to the $\mathcal{H}_\infty$ norm of the error transfer matrix.
- Extension to multi-agent systems with arbitrary homogeneous (heterogeneous) linear dynamics at the vertices.
- Combination of clustering method with edge removal.